

# $\ell_1$ -Regularized Linear Regression: Persistence and Oracle Inequalities

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## $\ell_1$ -regularized linear regression

- ▶ Random pair:  $(X, Y) \sim P$ , in  $\mathbb{R}^d \times \mathbb{R}$ .
- ▶  $n$  independent samples drawn from  $P$ :  
 $(X_1, Y_1), \dots, (X_n, Y_n)$ .
- ▶ Find  $\beta$  so linear function  $\langle X, \beta \rangle$  has small risk,

$$P\ell_\beta = P(\langle X, \beta \rangle - Y)^2.$$

Here,  $\ell_\beta(X, Y) = (\langle X, \beta \rangle - Y)^2$  is the quadratic loss of the linear prediction.

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**Example.**  $\ell_1$ -regularized least squares:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} P_n \ell_\beta + \rho_n \|\beta\|_{\ell_1^d},$$

$$\text{where } P_n \ell_\beta = \frac{1}{n} \sum_{i=1}^n (\langle X_i, \beta \rangle - Y_i)^2, \text{ and } \|\beta\|_{\ell_1^d} = \sum_{j=1}^d |\beta_j|.$$

## $\ell_1$ -regularized linear regression

**Example.**  $\ell_1$ -regularized least squares:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} P_n \ell_{\beta} + \rho_n \|\beta\|_{\ell_1^d} \quad ,$$

$$\text{where } P_n \ell_{\beta} = \frac{1}{n} \sum_{i=1}^n (\langle X_i, \beta \rangle - Y_i)^2, \text{ and } \|\beta\|_{\ell_1^d} = \sum_{j=1}^d |\beta_j|.$$

- ▶ Tends to select **sparse** solutions (few non-zero components  $\beta_j$ ).
- ▶ Useful, for example, if  $d \gg n$ .

## $\ell_1$ -regularized linear regression

**Example.**  $\ell_1$ -regularized least squares:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} P_n \ell_{\beta} + \rho_n \|\beta\|_{\ell_1^d} ,$$

**Example.**  $\ell_1$ -constrained least squares:

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1^d} \leq b_n} P_n \ell_{\beta} .$$

[Recall:  $\ell_{\beta}(X, Y) = (\langle X, \beta \rangle - Y)^2$ .]

## $\ell_1$ -regularized linear regression

**Example.**  $\ell_1$ -regularized least squares:

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**Example.**  $\ell_1$ -constrained least squares:

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1^d} \leq b_n} P_n \ell_{\beta} .$$

Some questions:

- **Prediction:** Does  $\hat{\beta}$  give accurate forecasts?  
e.g., How does  $P \ell_{\hat{\beta}}$  compare with  $P \ell_{\beta^*}$ ?

$$\text{Here, } \beta^* = \arg \min_{\|\beta\|_{\ell_1^d} \leq b_n} P_n \ell_{\beta} .$$

## $\ell_1$ -regularized linear regression

**Example.**  $\ell_1$ -regularized least squares:

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**Example.**  $\ell_1$ -constrained least squares:

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1^d} \leq b_n} P_n \ell_{\beta} .$$

Some questions:

- ▶ Does  $\hat{\beta}$  give accurate forecasts?  
e.g.,  $P \ell_{\hat{\beta}}$  versus  $P \ell_{\beta^*} = \min_{\|\beta\|_{\ell_1^d} \leq b_n} P \ell_{\beta}$  ?
- ▶ **Estimation:** Under assumptions on  $P$ , is  $\hat{\beta} \approx \beta^*$  correct?
- ▶ **Sparsity Pattern Estimation:** Under assumptions on  $P$ , are the non-zeros of  $\hat{\beta}$  correct?

## Outline of Talk

1. For  $\ell_1$ -constrained least squares, **bounds on  $P^{\ell_{\hat{\beta}}} - P^{\ell_{\beta^*}}$** .
  - ▶ **Persistence:** (Greenshtein and Ritov, 2004)  
For what  $d_n, b_n \rightarrow \infty$  does  $P^{\ell_{\hat{\beta}}} - P^{\ell_{\beta^*}} \rightarrow 0$ ?
  - ▶ **Convex Aggregation:** (Tsybakov, 2003)  
For  $b = 1$  (convex combinations of dictionary functions),  
what is rate of  $P^{\ell_{\hat{\beta}}} - P^{\ell_{\beta^*}}$ ?
2. For  $\ell_1$ -regularized least squares, **oracle inequalities**.
3. Proof ideas.

## $\ell_1$ -regularized linear regression

**Key Issue:**  $\ell_\beta$  is unbounded, so some key tools (e.g., concentration inequalities) cannot immediately be applied.

- ▶ For  $(X, Y)$  bounded,  $\ell_\beta$  can be bounded using  $\|\beta\|_{\ell_1^d}$ , but this gives loose prediction bounds.
- ▶ We use chaining to show that metric structures of  $\ell_1$ -constrained linear functions under  $P_n$  and  $P$  are similar.

## Main Results: Excess Risk

For  $\ell_1$ -constrained least squares,

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1^d} \leq b} P_n \ell_{\beta},$$

if  $X$  and  $Y$  have suitable tail behaviour then, with probability  $1 - \delta$ ,

$$P \ell_{\hat{\beta}} - P \ell_{\beta^*} \leq \frac{c \log^{\alpha}(nd)}{\delta^2} \min \left\{ \frac{b^2}{n} + \frac{d}{n}, \frac{b}{\sqrt{n}} \right\} 1 + \frac{b}{\sqrt{n}}.$$

- ▶ Small  $d$  regime:  $d/n$ .
- ▶ Large  $d$  regime:  $b/\sqrt{n}$ .

## Main Results: Excess Risk

For  $\ell_1$ -constrained least squares, with probability  $1 - \delta$ ,

$$P\ell_{\hat{\beta}} - P\ell_{\beta^*} \leq \frac{c \log^\alpha(nd)}{\delta^2} \min \left\{ \frac{b^2}{n} + \frac{d}{n}, \frac{b}{\sqrt{n}} \right\} \left( 1 + \frac{b}{\sqrt{n}} \right).$$

Conditions:

1.  $PY^2$  is bounded by a constant.
2.
  - ▶  $\|X\|_\infty$  bounded a.s.,
  - ▶  $X$  log concave and  $\max_j \|\langle X, e_j \rangle\|_{L_2} \leq c$ , or
  - ▶  $X$  log concave and isotropic.

## Application: Persistence

Consider  $\ell_1$ -constrained least squares,

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1} \leq b} P_n \ell_{\beta}.$$

Suppose that  $PY^2$  is bounded by a constant and tails of  $X$  decay nicely (e.g.,  $\|X\|_{\infty}$  bounded a.s. or  $X$  log concave and isotropic).

Then for increasing  $d_n$  and

$$b_n = o \left( \frac{\sqrt{n}}{\log^{3/2} n \log^{3/2}(nd_n)} \right),$$

$\ell_1$ -constrained least squares is persistent  
(i.e.,  $P\ell_{\hat{\beta}} - P\ell_{\beta^*} \rightarrow 0$ ).

## Application: Persistence

If  $PY^2$  is bounded and tails of  $X$  decay nicely, then  $\ell_1$ -constrained least squares is persistent provided that  $d_n$  is increasing and

$$b_n = o\left(\frac{\sqrt{n}}{\log^{3/2} n \log^{3/2}(nd_n)}\right).$$

Previous Results (Greenshtein and Ritov, 2004):

1.  $b_n = \omega(n^{1/2} / \log^{1/2} n)$  implies empirical minimization is not persistent for Gaussian  $(X, Y)$ .
2.  $b_n = o(n^{1/2} / \log^{1/2} n)$  implies empirical minimization is persistent for Gaussian  $(X, Y)$ .
3.  $b_n = o(n^{1/4} / \log^{1/4} n)$  implies empirical minimization is persistent under tail conditions on  $(X, Y)$ .

## Application: Convex Aggregation

Consider  $b = 1$ , so that the  $\ell_1$ -ball of radius  $b$  is the convex hull of a dictionary of  $d$  functions (the components of  $X$ ).

Tsybakov (2003) showed that, for any aggregation scheme  $\hat{\beta}$ , the [rate of convex aggregation](#) satisfies

$$P_{\ell_{\hat{\beta}}} - P_{\ell_{\beta^*}} = \Omega \left( \min \left\{ \frac{d}{n}, \frac{\log d}{n} \right\} \right).$$

For bounded, isotropic distributions, our result implies that this rate can be achieved, up to log factors, by [least squares](#) over the convex hull of the dictionary.

Previous positive results (Tsybakov, 2003; Bunea, Tsybakov and Wegkamp, 2006) involved complicated estimators.



## Outline of Talk

1. For  $\ell_1$ -constrained least squares, bounds on  $P^{\ell_{\hat{\beta}}} - P^{\ell_{\beta^*}}$ .
  - ▶ **Persistence:**  
For what  $d_n, b_n \rightarrow \infty$  does  $P^{\ell_{\hat{\beta}}} - P^{\ell_{\beta^*}} \rightarrow 0$ ?
  - ▶ **Convex Aggregation:**  
For  $b = 1$  (convex combinations of dictionary functions),  
what is rate of  $P^{\ell_{\hat{\beta}}} - P^{\ell_{\beta^*}}$ ?
2. For  $\ell_1$ -regularized least squares, oracle inequalities.
3. Proof ideas.

## Proof Ideas: 1. $\epsilon$ -equivalence of $P$ and $P_n$ structures

Define

$$G_\lambda = \frac{\lambda}{P(\ell_\beta - \ell_{\beta^*})}(\ell_\beta - \ell_{\beta^*}) : P(\ell_\beta - \ell_{\beta^*}) \geq \lambda \quad .$$

Then:

**E**  $\sup_{g \in G_\lambda} |P_n g - P g|$  is small

$\Rightarrow$  with high probability, for all  $\beta$  with  $P(\ell_\beta - \ell_{\beta^*}) \geq \lambda$ ,

$$(1 - \epsilon)P(\ell_\beta - \ell_{\beta^*}) \leq P_n(\ell_\beta - \ell_{\beta^*}) \leq (1 + \epsilon)P(\ell_\beta - \ell_{\beta^*})$$

$\Rightarrow P(\ell_{\hat{\beta}} - \ell_{\beta^*}) \leq \lambda$ , where  $\hat{\beta} = \arg \min_\beta P_n \ell_\beta$ .



## Proof Ideas: 2. Symmetrization, subgaussian tails

## Proof Ideas: 3. Chaining

For a subgaussian process  $\{Z_t\}$  indexed by a metric space  $(T, d)$ , and for  $t_0 \in T$ ,

$$\mathbf{E} \sup_{t \in T} |Z_t - Z_{t_0}| \leq c \mathcal{D}(T, d) = c \int_0^{\text{diam}(T, d)} \sqrt{\frac{\log N(\epsilon, T, d)}{\log 2}} d\epsilon,$$

where  $N(\epsilon, T, d)$  is the  $\epsilon$  covering number of  $T$ .

## Proof Ideas: 4. Bounding the Entropy Integral

It suffices to calculate the entropy integral  $\mathcal{D}(\sqrt{\lambda}D \cap 2bB_1^d, d)$ .  
We can approximate this by

$$\mathcal{D}(\sqrt{\lambda}D \cap 2bB_1^d, d) \leq \min \mathcal{D}(\sqrt{\lambda}D, d), \mathcal{D}(2bB_1^d, d) \quad .$$

This leads to:

$$P_{\ell_{\hat{\beta}}} - P_{\ell_{\beta^*}} \leq \frac{c \log^\alpha(nd)}{\delta^2} \min \left( \frac{b^2}{n} + \frac{d}{n}, \frac{b}{\sqrt{n}} \right) \left( 1 + \frac{b}{\sqrt{n}} \right) \quad .$$

## Proof Ideas: 5. Oracle Inequalities

We get an isomorphic condition on  $\{\ell_\beta - \ell_{\beta^*}\}$ ,

$$\frac{1}{2}P_n(\ell_\beta - \ell_{\beta^*}) - \epsilon_n \leq P(\ell_\beta - \ell_{\beta^*}) \leq 2P_n(\ell_\beta - \ell_{\beta^*}) + \epsilon_n,$$

and this implies that  $\hat{\beta} = \arg \min_{\beta} (P_n \ell_\beta + c\epsilon_n)$  has

$$P\ell_{\hat{\beta}} \leq \inf_{\beta} P\ell_{\beta} + c'\epsilon_n .$$

This leads to oracle inequality: For  $\ell_1$ -regularized least squares,

$$\hat{\beta} = \arg \min_{\beta} P_n \ell_{\beta} + \rho_n \|\beta\|_{\ell_1^{d_n}} ,$$

with probability at least  $1 - o(1)$ ,

$$P\ell_{\hat{\beta}} \leq \inf_{\beta} P\ell_{\beta} + c\rho_n (1 + \|\beta\|_{\ell_1^{d_n}}) .$$

## Outline of Talk

1. For  $\ell_1$ -constrained least squares,

$$P\ell_{\hat{\beta}} - P\ell_{\beta^*} \leq \frac{c \log^\alpha(nd)}{\delta^2} \min \left[ \frac{b^2}{n} + \frac{d}{n}, \frac{b}{\sqrt{n}} \right] \left( 1 + \frac{b}{\sqrt{n}} \right) .$$

- **Persistence:**

If  $b_n = \tilde{o}(\sqrt{n})$ , then  $P\ell_{\hat{\beta}} - P\ell_{\beta^*} \rightarrow 0$ .

- **Convex Aggregation:**

Empirical risk minimization gives optimal rate (up to log factors):  $\tilde{O} \left( \min(d/n, \sqrt{\frac{d}{n \log d/n}}) \right)$  .

2. For  $\ell_1$ -regularized least squares, oracle inequalities.
3. Proof ideas: subgaussian Rademacher process.